Mathematics 222B Lecture 16 Notes

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1 Regularity Estimates for Variable-Coefficient Wave Equations

1.1 Well-posedness of the initial value problem for variable-coefficient wave equations

Today, we are interested in a concrete goal. We will be studying **variable-coefficient wave equations**, PDEs of the form

$$P\phi = \partial_\mu(g^{\mu,\nu}\partial_\nu\phi) + b^\mu\partial_\mu\phi + c\phi,$$

where the key assumption is that g is a symmetric matrix with signature $(-,+,+\dots,+)$. The example we should keep in mind is $g = \operatorname{diag}(-1,1,1,\dots,1)$, $b=0$, $c=0$; this makes $P = \Box$. We are solving the initial value problem

$$\begin{cases} P\phi = f & \text{in } (0,\infty)_t \times \mathbb{R}^d \\ (\phi, \partial_t\phi)|_{t=0} = (g,h) & \text{on } \{t=0\} \times \mathbb{R}^d. \end{cases}$$

We further assume that $g^{\mu,\nu}, b^\mu, c$ are bounded with bounded derivatives of all orders. We also assume a restricted form of g (which we will later show is not much of a restriction): $g^{tt} = -1$ and $g^{t,x^j} = 0$. This means that if we write g as a matrix,

$$g = \begin{bmatrix} -1 & 0_{1\times d} \\ 0_{d\times 1} & \overline{g}, \end{bmatrix}$$

where $\overline{g}$ is **uniformly elliptic** ($\overline{g} \succ \lambda I$).

Our concrete goal is to prove the following theorem:

Theorem 1.1. *The initial value problem is well-posed in $H^k \times H^{k-1}$ for all $k \in \mathbb{Z}$. That is,*

(i) (Existence) Given $(g,h) \in H^k \times H^{k-1}$ and $f \in L^1_t(H^{k-1})$, there exists a solution ϕ to the initial value problem in the class $C_t(\mathcal{H}^k)$.

(ii) (Uniqueness) The solution ϕ in $C_t(\mathcal{H}^k)$ to the initial value problem with (f, g, h) as in (i) is unique.

(iii) (Continuous dependence)

$$\sup_t \|\phi(\phi, \partial_t \phi)\| \leq C_k(\|(g, h)\|_{\mathcal{H}^k} + \|f\|_{L^1_t(H^{k-1})}).$$

Here, $\mathcal{H}^k = H^k \times H^{k-1}$, and by $\phi \in C_t(I; \mathcal{H}^k)$, we mean that $\phi \in C_t(I; H^k)$ and $\partial_t \phi \in C_t(I; H^{k-1})$.

We will use the convention that $\mathbb{R}^{1+d} = \{(t = x^0, x^1, \dots, x^d)\}$. The Greek indices μ, ν will range from $0, 1, \dots, d$, while the indices j, k, ℓ will range from $1, \dots, d$. We will also denote $g^{t,t} = g^{0,0}, g^{t,x^j} = g^{0,j}$.

Remark 1.1. The problem is **time reversible**. If we send $t \mapsto -t$, the equation is essentially unchanged.

The reference for this topic is chapters 6-7 of Ringström's book.

1.2 Energy inequality for P

The basic ingredient in this proof is an energy inequality for P. Suppose $P\phi = f$. The idea is to multiply the equation by $\partial_t \phi$ and "integrate by parts." Why should we multiply by $\partial_t \phi$ instead of ϕ? This is a generalization of what we do in the classical wave equation, and we will be able to give a more insightful answer to this once we discuss calculus of variations for problems of this type. The key observation is this integration by parts idea, but in divergence form:

$$\begin{aligned}
\partial_\mu(g^{\mu,\nu}\partial_\nu \phi)\partial_t \phi &= -\partial_t^2 \phi \partial_t \phi + \partial_j(\overline{g}^{j,k}\partial_k \phi)\partial_t \phi \\
&= \partial_t \left(-\frac{1}{2}(\partial_t \phi)^2\right) + \partial_j(\overline{g}^{j,k}\partial_k \phi \partial_t \phi) - \overline{g}^{j,k}\partial_k \phi \partial_j \partial_t \phi
\end{aligned}$$

Since g is symmetric, this last term can be written as $-\overline{g}^{j,k}\partial_t(\partial_k \phi \partial_j \phi)$ by symmetrizing. Moving the ∂_t to the outside, we get

$$= \partial_t \left(-\frac{1}{2}(\partial_t \phi)^2\right) - \frac{1}{2}\overline{g}^{j,k}\partial_j \phi \partial_k \phi + \partial_j(\overline{g}^{j,k}\partial_k \phi \partial_t \phi) + \frac{1}{2}\partial_t \overline{g}^{j,k}\partial_j \phi \partial_k \phi.$$

This form is nice because the terms that have the maximum number of derivatives are all in divergence form, while the terms that don't have the maximum number of derivatives are not in divergence form.

Integrate this on $(t_0, t_1) \times \mathbb{R}^d =: R_{t_0}^{t_1}$ (assuming the boundary term vanishes):

$$\iint_{R_{t_0}^{t_1}} \partial_\mu(g^{\mu,\nu}\partial \nu \phi)\partial_t \phi - \frac{1}{2}\iint_{R_{t_0}^{t_1}} \partial_t \overline{g}^{j,k}\partial_j \phi \partial_k \phi$$

$$= -\int_{\Sigma_{t_1}} \frac{1}{2}((\partial_t\phi)^2 + \overline{g}^{j,k}\partial_j\phi\partial_k\phi) + \int_{\Sigma_{t_0}} \frac{1}{2}((\partial_t\phi)^2 + \overline{g}^{j,k}\partial_j\phi\partial_k\phi$$
$$+ \underbrace{\lim_{R\to\infty}\int_{t_0}^{t_1}\int_{\partial B_R} \nu_j(\overline{g}^{j,k}\partial_k\phi\partial_t\phi)\, dA\, dt}_{=0},$$

where $\Sigma_t = \{t\} \times \mathbb{R}^d$.

Denote $\vec{\phi} = (\phi, \partial_t\phi)$, so $(\phi, \partial_t\phi) \in \mathcal{H}^k$ if and only if $\vec{\phi} \in C_t(\mathcal{H}^k)$.

Lemma 1.1. *For $\phi \in C_t(\mathcal{H}^1)$,*

$$\sup_{t\in[0,T]} \|\vec{\phi}\|_{\mathcal{H}^k} \le C_T \left(\|\vec{\phi}(0)\|_{\mathcal{H}^1} + \int_0^T \|P\phi\|_{L^2}\, dt \right).$$

Proof. We may assume without loss of generality that $\phi \in C^\infty(\overline{R_0^T})$ and $\phi(t,\cdot)$ has compact support for each $t \in [0,T]$. By the computation above, if

$$E[\phi](t) = \frac{1}{2}\int_{\Sigma_t} (\partial_t\phi)^2 + \overline{g}^{j,k}\partial_j\phi\partial_k\phi\, dx,$$

then

$$\mathbb{E}[\phi](t_1) = \mathbb{E}[\phi](0) - \iint_{R_0^{t_1}} \partial_\mu(g^{\mu,\nu}\partial_\nu\phi) + \frac{1}{2}\iint_{R_0^{t_1}} \partial_t\overline{g}^{j,k}\partial_j\partial_k\phi.$$

(Note that $\lim_{R\to\infty}\int_{\partial B_R} = 0$ thanks to the support assumption. Now

$$\partial_\mu(g^{\mu,\nu}\partial_\nu\phi) = P\phi - b^\mu\partial_\mu\phi - c\phi,$$

which tells us that

$$\mathbb{E}[\phi](t_1) = E[\phi](0) + \iint_{R_0^t} P\phi\partial_t\phi\, dx\, dt + \iint_{R_0^t} (b^\mu\partial_\mu\phi\partial_t\phi + c\phi\partial_t\phi + \partial_t\overline{g}^{j,k}\partial_j\phi\partial_k\phi)\, dx\, dt.$$

Call the error

$$\mathcal{E}_0^t = \iint_{R_0^t} |b^\mu\partial_\mu\phi\partial_t\phi + c\phi\partial_t\phi + \partial_t\overline{g}^{j,k}\partial_j\phi\partial_k\phi|\, dx\, dt.$$

We get an inequality:

$$\sup_{t_1\in[0,T]} E[\phi](t_1) \le E[\phi](0) + \sup_{t\in[0,T} \left| \iint_{R_0^t} P\phi\partial_t\phi\, dx\, dt \right| + \mathcal{E}_0^T.$$

Note that $E[\phi] \ge \frac{1}{2}\int (\partial_t\phi)^2(t)\, dx \ge \frac{\lambda}{2}\int |D_t\phi|^2(t)\, dx$. Using the fundamental theorem of calculus,

$$\int |\phi|^2(t)\, dx = \int_0^t \int \partial\phi\phi\, dx\, dt' + \int |\phi|^2(0)\, dx$$

Using Cauchy-Schwarz,

$$\leq 2 \int E(t')^{1/2} \left(\int |\phi|^2(t')\, dx \right)^{1/2} dt' + \int |\phi|^2(0)\, dx.$$

Skipping a few steps, we get

$$\sup_{t\in[0,T]} \int |\phi|^2(t)\, dt \leq \int |\phi|^2(0)\, dx + CT \sup_{t\in[0,T]} E(t).$$

The point here is that

$$\sup_{t\in[0,T]} \|\vec{\phi}\|_{\mathcal{H}^1} \leq C_T \left(\|\vec{\phi}(0)\|^2_{\mathcal{H}^1} + \sup_{t\in[0,T]} \left| \iint_{R_0^T} P\phi \partial_t \phi \, dx\, dt \right| + \mathcal{E}_0^T \right).$$

If we use Cauchy-Schwarz, we get

$$\begin{aligned} \sup_{t\in[0,T]} \left| \iint_{R_0^t} P\phi\partial_t\phi\, dx\, dt \right| &\leq \int_0^T \|P\phi(t)\|_{L^2} \|\partial_t \phi\|_{L^2}\, dt \\ &\leq C \int_0^T \|P\phi(t)\|_{L^2} E[\phi]^{1/2}\, dt \\ &\leq \int_0^T \|P\phi(t)\|_{L^2} dt \sup_{[0,T]} E[\phi]^{1/2} \end{aligned}$$

We can use Cauchy-Schwarz to absorb the energy term to the left hand side, since $E[\phi] \leq C \int (\partial_t \phi)^2 + (D_x \phi)^2$. We get

$$\sup_{t\in[0,t_1]} \|\vec{\phi}\|^2_{\mathcal{H}^1} \leq C_T \left(\|\vec{\phi}(0)\|^2_{\mathcal{H}^1} + \int_0^T \|P\phi\|_{L^2}\, dt + \int_0^{t_1} \|\phi(t)\|^2_{\mathcal{H}^1}\, dt \right).$$

This means that if we let $\mathcal{D}(t_1)$ be the left hand side and $\mathcal{D}_0$ be the first two terms on the right hand side, we get

$$\mathcal{D}(t_1) \leq \mathcal{D}_0 + \int_0^{t_1} \mathcal{D}(t)\, dt.$$

Using Grönwall's inequality, we get

$$\mathcal{D}(t) \leq \mathcal{D}_0 \exp\left(\int_0^t dt' \right) \leq \mathcal{D}_0 e^T.$$

This finishes the proof. □

1.3 Further regularity estimates for existence and uniqueness

We want to study something like $P : C_t(\mathcal{H}^k) \to L^1_t(H^{k-1})$. This means that we should look at the adjoint $P^* : C_t(H^{-(k-1)}) \to L^1_t(H^{-k})$. The dual problem here includes negative Sobolev spaces.

Lemma 1.2. *For any $k \in \mathbb{Z}$ and $\phi \in C_t(\mathcal{H}^{1+k}) \cap C^\infty_{t,x}$,*

$$\sup_{t\in[0,T]} \|\vec{\phi}(t)\|_{\mathcal{H}^{1+k}} \leq C_{T,k} \left(\|\vec{\phi}(0)\|_{\mathcal{H}^{1+k}} + \int_0^T \|P\phi\|_{H^k}\, dt \right).$$

The positive regularities will give us uniqueness for the initial value problem. The negative regularities will give us existence.

Proof. For $k > 0$, we commute the equation with D^α for $|\alpha| \leq k$. Then apply the previous lemma and Grönwall's inequality. (This technique is very similar to our previous proof of higher elliptic regularity bounds. However, we don't need to use a difference quotient.)

For $k < 0$, we work with $\Phi = (1-\Delta)^{-|k|}\phi$. (This means that we want to look at the solution to the elliptic problem $(1-\Delta)^{|k|}\Phi = \phi$ in $\mathbb{R}^d$. Another way to write this is $\widehat{\Phi} = (1-|\xi|^2)^{-|k|}\widehat{\phi}$.) We do this so that we don't have to deal with negative Sobolev spaces; we can study an operator that commutes well with P and use positive Sobolev spaces, instead. The key thing to notice is that $(1-\Delta)^{-\ell} : H^s \to H^{s+2\ell}$. We also use the following:

Lemma 1.3. *For any $s \in \mathbb{R}$, the H^s norm has the Fourier characterization*

$$\begin{aligned} \|v\|_{H^s} &= \|(1+|\xi|^2)^{s/2}\widehat{v}\|^2_{L^2_\xi} \\ &= \|(1-\Delta)^{s/2} v\|^2_{L^2}. \end{aligned}$$

When $s \in 2\mathbb{Z}$, this agrees with our sense of derivatives.

We want to compute

$$\begin{aligned} \|P\Phi\|^2_{H^{|k|}} &= \|(1+\|xi|^2)^{|k|/2}\widehat{P\Phi}\|^2_{L^2} \\ &= \langle (1+|\xi|^2)^{|k|/2}\widehat{P\Phi}, (1+|\xi|^2)^{|k|/2}\widehat{P\Phi}\rangle \\ &= \langle (1+|\xi|^2)^{|k|/2}\widehat{P\Phi}, \widehat{P\Phi}\rangle \\ &= \langle (1-\Delta)^{|k|}P\Phi, P\Phi\rangle. \end{aligned}$$

Now observe that

$$\begin{aligned} (1-\Delta)^{|k|}P\Phi &= P((1-\Delta)^{|k|}\Phi) + [(1-\Delta)^{|k|}, P]\Phi \\ &= P\phi + \underbrace{[(1-\Delta)^{|k|}, P]}_{\text{order } 2|k|+2-1}\Phi. \end{aligned}$$

This tells us that

$$\begin{aligned} \|\vec{\Phi}(t)\|_{\mathcal{H}^{1+|k|}} &= \|\vec{\phi}(t)\|_{\mathcal{H}^{1+|k|-2|k|}} \\ &= \|\widehat{\phi}(t)\|_{\mathcal{H}^{1+k}} \end{aligned}$$

for $k < 0$. $\square$